6. L^p SPACES — PART I

Last week: ☐ Convergence theorems for integrals

- Monotone convergence theorem

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- Fatou's lemma

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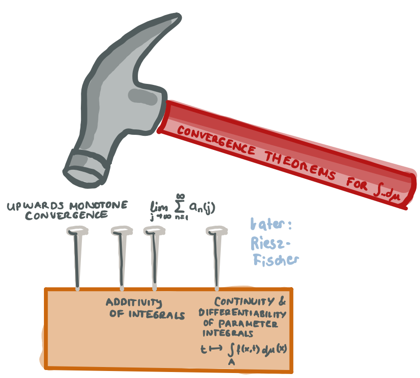
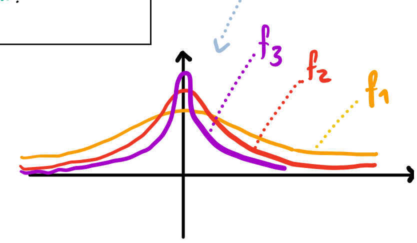
- Dominated convergence theorem



> **Theorem 5.14** (Dominated convergence)
> Let (X, Γ, μ) be a measure space and $A \in \Gamma$.
> Assume that $f, f_k : A \to [-\infty, \infty]$ are Γ-measurable functions such that
> $$\lim_{k\to\infty} f_k(x) = f(x) \quad \text{for } \mu \text{ a.e. } x \in A.$$
> If there is $g \in \mathcal{L}^1(A; \mu)$ such that, for all $k \in \mathbb{N}$,
> $$|f_k(x)| \leq g(x) \quad \text{for } \mu \text{ a.e. } x \in A$$
> Then $f, f_k \in \mathcal{L}^1(A; \mu)$, $k \in \mathbb{N}$, and
> $$\lim_{k\to\infty} \int_A f_k \, d\mu = \int_A f \, d\mu.$$

integrable dominating function prevents $\left|\int_A f \, d\mu\right|$ from getting too small

Example where neither monotone nor dominated convergence is applicable



Today: ☐ $\mathcal{L}^p(A; \mu)$ for $1 \leq p \leq \infty$

☐ $L^p(A; \mu)$ for $1 \leq p \leq \infty$

☐ Hölder's & Minkowski's inequality

- Let (X,Γ,μ) be a measure space and $A\in\Gamma$. (6.1.2)
  Recall that $f: A\to[-\infty,\infty]$ is μ integrable
  (denoted $f\in\mathcal{L}^1(A;\mu)$)

  if and only if

  f is Γ-measurable and $\int_A |f|\,d\mu<\infty$.

To measure „difference" of functions f,g: „size" of $\lvert f-g\rvert$? $\int_A \lvert f-g\rvert\,d\mu$? Putting different weights on the error $\lvert f-g\rvert$?

$\mathcal{L}^p(A;\mu)$ FOR $1\le p<\infty$

- For any Γ-measurable $f: A\to[-\infty,\infty]$, the integral

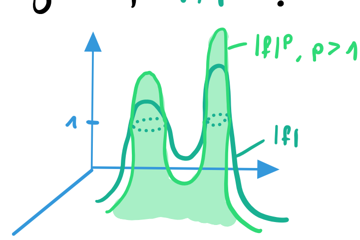
$$\int_A |f|^p\,d\mu$$

  exists for any $1\le p<\infty$ and one can ask similarly whether it is finite or not.
  The answer typically depends on the exponent p!

  Large integrability exponents $p>1$ put more weight on large values of $|f(x)|$:

  - if $|f(x)|<1$, then $|f(x)|^p<|f(x)|$
  - if $|f(x)|=1$, then $|f(x)|^p=|f(x)|$
  - if $|f(x)|>1$, then $|f(x)|^p>|f(x)|$

- If $0<\mu(A)<\infty$, we can consider $\frac{1}{\mu(A)}\int_A |f|^p\,d\mu$ as „average height of $|f|^p$".



Def. 6.1 (p-normi)

Let (X, Γ, μ) be a measure space, $A \in \Gamma$, and $1 \leq p < \infty$. The p-norm of a Γ-measurable function $f: A \to [-\infty, \infty]$ is

$$\|f\|_p := \|f\|_{L^p(A)} := \left(\int_A |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

- The exponent "$\frac{1}{p}$" ensures homogeneity under scalar muliplication:

$$\|\lambda f\|_p = \left(\int_A |\lambda|^p |f|^p \, d\mu \right)^{\frac{1}{p}} = |\lambda| \, \|f\|_p ,$$

  but for Γ-measurable $f: A \to [-\infty, \infty]$:

$$\|f\|_p < \infty \iff \int_A |f|^p d\mu < \infty \iff |f|^p \text{ is } \mu\text{-integrable}.$$

- It can happen that $\|f\|_p = \infty$. The class of Γ-measurable functions for which $\|\cdot\|_p$ is finite is denoted with a special symbol (it depends on p and on the measure μ):

Def. 6.2

Let (X, Γ, μ) be a measure space, $A \in \Gamma$, and $1 \leq p < \infty$. Then we define

$$\mathcal{L}^p(A; \mu) := \{ f: A \to [-\infty, \infty] \ \Gamma\text{-measurable and } \|f\|_p < \infty \}.$$

- In general, neither for $p<q$ neither $\mathcal{L}^p(A;\mu) \subset \mathcal{L}^q(A;\mu)$ (6.1.4) nor $\mathcal{L}^q(A;\mu) \subset \mathcal{L}^p(A;\mu)$ holds. (Consider e.g., $f(x) = 1/|x|^\alpha$ for suitable α on $A \subset X = \mathbb{R}$ with a suitable measure μ ...)

  However:

Lemma 6.3

Let (X, Γ, μ) be a measure space, $A \in \Gamma$, and $1 \leq p < q < \infty$. If $\mu(A) < \infty$, then

$$\mathcal{L}^q(A;\mu) \subset \mathcal{L}^p(A;\mu).$$

Proof. [Exercise]

Heuristics: μ-integrability of a measurable function f on A can fail essentially for two reasons: because the function grows too much or the measure grows too much (or both).

If $\mu(A) < \infty$, the second problem cannot occur, and if the first problem is under control for $|f|^q$, then also for $|f|^p$ with $p<q$.

$\mathcal{L}^p(A;\mu)$ FOR $p=\infty$

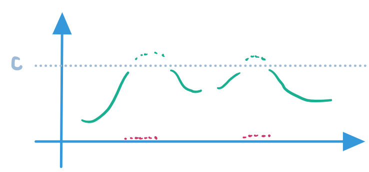
Def. 6.4

Let (X,Γ,μ) be a measure space and $A\in\Gamma$.
For a Γ-measurable function $f: A\to[-\infty,\infty]$, we denote

$$\|f\|_\infty := \|f\|_{L^\infty(A)} := \operatorname*{ess\,sup}_{x\in A}|f(x)| := \inf\{c\geq 0: |f(x)|\leq c \ \ \mu \text{ a.e. } x\in A\}$$

and

$$\mathcal{L}^\infty(A;\mu) = \{f: A\to[-\infty,\infty]: \Gamma\text{-measurable with } \|f\|_\infty<\infty\}$$



- If $f\in\mathcal{L}^\infty(A;\mu)$, then not necessarily $|f(x)|\leq\|f\|_\infty$ for all $x\in A$, but $\mu(\{x\in A: |f(x)|>\|f\|_\infty\})=0$ and hence $|f(x)|\leq\|f\|_\infty$ for μ a.e. $x\in A$.
  (This requires a short justification since $\|\cdot\|_\infty$ is defined using the infimum.)

- With this information, we can prove Lemma 6.3 for $q=\infty$:

Lemma 6.5

Let (X,Γ,μ) be a measure space, $A\in\Gamma$, and $1\leq p<\infty$.
If $\mu(A)<\infty$, then $\mathcal{L}^\infty(A;\mu)\subset\mathcal{L}^p(A;\mu)$.

Proof. $f\in\mathcal{L}^\infty(A;\mu)$ satisfies $\int_A |f|^p d\mu \leq \int_A \|f\|_\infty^p \, d\mu = \|f\|_\infty^p\,\mu(A)$,
so if $\mu(A)<\infty$, it follows $f\in\mathcal{L}^p(A;\mu)$ for every $p\geq 1$. $\square$

This fails if $\mu(A)=\infty$ (Ex: constant function $f\equiv 1\in\mathcal{L}^\infty(A;\mu)\setminus\mathcal{L}^p(A;\mu)$)

$L^p(A;\mu)$ FOR $1 \leq p \leq \infty$

- $\|\cdot\|_p$ is not a norm on $\mathcal{L}^p(A;\mu)$!

Recall: A function $\|\cdot\|: V \to [0,+\infty[$ on a $\mathbb{R}$-vector space V is a norm on V if

(i) $\|v\| = 0 \iff v = 0$ [positive definite]

(ii) $\|\lambda v\| = |\lambda| \, \|v\|$ for all $v \in V, \lambda \in \mathbb{R}$

(iii) $\|v+w\| \leq \|v\| + \|w\|$ for all $v, w \in V$ [triangle inequality]

A familiar example: $(\mathbb{R}^n, \|\cdot\|)$ with $\|x\| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$.

- Problem 1 (vector space structure)

If $f, g \in \mathcal{L}^p(A;\mu)$, then $|f|^p, |g|^p \in \mathcal{L}^1(A;\mu)$ and Theorem 5.11 (i) implies that $|f(x)| < \infty$ and $|g(x)| < \infty$ μ a.e. in A. Hence $f+g$ is only defined μ a.e. in A, namely outside

$$\{x \in A: f(x) = \infty, g(x) = -\infty\} \cup \{x \in A: f(x) = -\infty, g(x) = \infty\}.$$

- Problem 2 (positive definiteness of norm)

By Theorem 5.11 (ii):

$$\|f\|_p = 0 \Rightarrow |f(x)|^p = 0 \ \mu \text{ a.e. } x \in A \Rightarrow f = 0 \ \mu \text{ a.e.},$$

but not necessarily everywhere.

- We would like to modify $\mathcal{L}^p(A;\mu)$ so as to obtain a normed space $(L^p(A;\mu), \|\cdot\|_p)$.
  This will be a good setting to do functional analysis:
  A norm allows to study how close to elements in the space are from each other, whether a sequence of elements converges to a limit etc. Since it induces a metric via $d(v,w) := \|v-w\|$, but normed space have more structure than arbitrary metric spaces since they are built on vector spaces that allow to add and rescale elements.

- $(\mathcal{L}^p(A;\mu), \|\cdot\|_p)$ will become a normed space if we identify functions that agree μ almost everywhere:

Def. 6.6
Let (X, Γ, μ) be a measure space, $A \in \Gamma$, and $1 \leq p \leq \infty$. Then we define an equivalence relation on $\mathcal{L}^p(A;\mu)$ by setting, for all $f, g \in \mathcal{L}^p(A;\mu)$,

$$f \sim g \iff f(x) = g(x) \text{ for } \mu \text{ a.e. } x \in A.$$

Recall: a binary relation $\sim$ on a set Y is an equivalence relation if it is:
(i) reflexive $\quad x \sim x$
(ii) symmetric $\quad x \sim y \iff y \sim x$
(iii) transitive $\quad x \sim y, y \sim z \Rightarrow x \sim z$.
(where $x, y, z \in Y$)

Def. 6.7

Let (X, Γ, μ) be a measure space, $A \in \Gamma$, and $1 \leq p \leq \infty$. For $f \in \mathcal{L}^p(A;\mu)$, we define the equivalence class

$$[f] = \{g \in \mathcal{L}^p(A;\mu) : g \sim f\} = \{g \in \mathcal{L}^p(A;\mu) : g = f \ \mu \text{ a.e. on } A\}.$$

The L^p - space is the quotient space

$$L^p(A;\mu) := \mathcal{L}^p(A;\mu) / \sim \ = \{[f] : f \in \mathcal{L}^p(A)\}$$

- The elements of $L^p(A;\mu)$ are not functions, but equivalence classes of functions. Nonetheless one often writes by abuse of notation things like „$\|f\|_p$ for $f \in L^p(A;\mu)$".

Def. 6.8

Let (X, Γ, μ) be a measure space, $A \in \Gamma$, and $1 \leq p \leq \infty$. For $[f] \in L^p(A;\mu)$, we define

$$\|[f]\|_p := \|f\|_p .$$

- $\|[f]\|_p$ is well defined: if $f \sim g$, then $f = g$ μ a.e. on A, and thus $\|f\|_p = \left(\int_A |f|^p d\mu\right)^{\frac{1}{p}} = \left(\int_A |g|^p d\mu\right)^{\frac{1}{p}} = \|g\|_p$.
- Is $(L^p(A;\mu), \|\cdot\|_p)$ a normed space?
  - Positive definiteness is now OK by definition:
    $\|[f]\|_p = 0 \Rightarrow f = 0 \ \mu$ a.e. $\Rightarrow [f] = [0]$
  - Trickier: $[f], [g] \in L^p \overset{?}{\Rightarrow} [f]+[g] \in L^p$, triangle inequality for $\|\cdot\|_p$?

HÖLDER'S INEQUALITY

- Why don't we consider $p<1$?
  Because of the triangle inequality!

  Example Consider $p=\frac{1}{2}$, $f=\chi_{[0,1]}$, $g=\chi_{]1,2]}$.
  Then

  $$\left(\int_{\mathbb{R}} |f+g|^{\frac{1}{2}}\, dm_1\right)^2 = m_1([0,2])^2 = 4 > 2 = \left(\int_{\mathbb{R}} |f|^{\frac{1}{2}}\, dm_1\right)^2 + \left(\int_{\mathbb{R}} |g|^{\frac{1}{2}}\, dm_1\right)^2.$$

- If $p \geq 1$, the situation looks better, thanks to the following inequality, which will eventually allow us to prove a triangle inequality for $\|\cdot\|_p$ if $p \geq 1$.

Theorem 6.9 (Hölderin epäyhtälö)
Let (X,Γ,μ) be a measure space, $A\in\Gamma$, and let $p,q\in[1,\infty]$ be conjugate Hölder exponents, that is, $\frac{1}{p}+\frac{1}{q}=1$. Then, if $f\in\mathcal{L}^p(A;\mu)$ and $g\in\mathcal{L}^q(A;\mu)$, we have

$$fg\in\mathcal{L}^1(A;\mu)$$

and

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q. \quad \text{[Hölder's inequality]}$$

- Why is this relevant for $\|\cdot\|_p$ - triangle inequality when $p>1$?

  Because in that case we can write for $p,q\in\mathcal{L}^p(A;\mu)$:

  $$\|f+g\|_p^p = \int_A |f+g|^p\, d\mu = \int_A |f+g|\,|f+g|^{p-1}\, d\mu \leq \overbrace{\int_A |f|\,|f+g|^{p-1}\, d\mu}^{\leq ?} + \overbrace{\int_A |g|\,|f+g|^{p-1}\, d\mu}^{\leq ?},$$

  where $|f|,|g|\in\mathcal{L}^p(A;\mu)$ ... [$\to$ see proof of Theorem 6.10].

- For $p, q \in]1,\infty[$, Hölder's inequality reads
$$\int_A |fg| \, d\mu \leq \left(\int_A |f|^p \, d\mu \right)^{\frac{1}{p}} \left(\int_A |g|^q \, d\mu \right)^{\frac{1}{q}}$$
- $p = q = 2$ are conjugate exponents: $\frac{1}{2} + \frac{1}{2} = 1$. In this case, Hölder's inequality is also known as Cauchy-Schwarz inequality.
- $p = 1$, $q = \infty$ are conjugate exponents: $\frac{1}{1} + \frac{1}{\infty} = 1 + 0 = 1$

Proof of Theorem 6.9.

- If $f \in \mathcal{L}^p(A;\mu)$ and $g \in \mathcal{L}^q(A;\mu)$, they are both Γ-measurable.
- It follows that fg is Γ-measurable.
- Hence, to show $fg \in \mathcal{L}^1(A;\mu)$, it suffices to prove $\|fg\|_1 < \infty$, which will follow from the inequality

  $(*) \qquad \|fg\|_1 \leq \|f\|_p \, \|g\|_q ,$

  so we concentrate on proving that, assuming that $f \in \mathcal{L}^p(A;\mu)$, $g \in \mathcal{L}^q(A;\mu)$.

  - Proof of $(*)$ for $p = 1$ and $q = \infty$

    $|f(x) g(x)| \leq |f(x)| \, \|g\|_\infty$ for μ a.e. $x \in A$ (see below Def. 6.4),
    hence
    $$\|fg\|_1 = \int_A |fg| \, d\mu \leq \|g\|_\infty \int_A |f| \, d\mu = \|g\|_\infty \|f\|_1 .$$

Proof of (∗) for $p, q \in]1,\infty[$ with $\frac{1}{p}+\frac{1}{q}=1$

Case 1: $\|f\|_p = 0$ or $\|g\|_q = 0$

In this case, we have $f=0$ μ a.e. on A or $g=0$ μ a.e. on A, and thus $\|fg\|_1 = 0$

(recall the convention $0\cdot\infty = 0$ in this context).

Then (∗) trivially holds true: $0 \leq \ldots$

Case 2: $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$

By assumption $0 < \|f\|_p, \|g\|_q < \infty$, and the following inequality is equivalent to (∗) under our assumptions:

$$(**) \qquad \frac{1}{\|f\|_p \, \|g\|_q} \int_A |fg| \, d\mu \leq \frac{1}{p} + \frac{1}{q}.$$

Inequality (∗∗) (and thus (∗)) can be proven as follows:

$$\frac{1}{\|f\|_p \, \|g\|_q} \int_A |f||g| \, d\mu = \int_A \underbrace{\frac{|f(x)|}{\|f\|_p}}_{=:a} \underbrace{\frac{|g(x)|}{\|g\|_q}}_{=:b} \, d\mu(x)$$

[Young's inequality] → exercise, using $\frac{1}{p}+\frac{1}{q}=1$

$$\leq \int_A \frac{1}{p} a^p + \frac{1}{q} b^q \, d\mu(x)$$

$$= \frac{1}{p} \frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q} \frac{\|g\|_q^q}{\|g\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q}$$

□

MINKOWSKI'S INEQUALITY

Goal: $(L^p(A;\mu), \|\cdot\|_p)$ is a normed space for $1 \le p \le \infty$.

> **Theorem 6.10** (Minkowskin epäyhtälö)
> Let (X, Γ, μ) be a measure space, $A \in \Gamma$, and let $p \in [1,\infty]$. If $f, g \in L^p(A;\mu)$, then also $f+g \in L^p(A;\mu)$ and
> $$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

- Recall that elements $f \in L^p(A;\mu)$ are actually equivalence classes $[f]$. We define $[f]+[g] := [f+g]$ with the following convention:

  If f and g are Γ-measurable functions with values in the extended real line $[-\infty,\infty]$, then $f(x)+g(x)$ is a priori only defined outside the set $B := \{x \in A : f(x) = \infty, g(x) = -\infty\} \cup \{x \in A : f(x) = -\infty, g(x) = \infty\}$, but it can be extended to a Γ-measurable function $A \to [-\infty,\infty]$.

  Since $|f(x)|, |g(x)| < \infty$ μ a.e. $x \in A$ if $f, g \in \mathcal{L}^p(A;\mu)$, all Γ-measurable extensions of $f+g$ for $f, g \in \mathcal{L}^p(A;\mu)$ agree μ a.e. on A (namely outside the null set B). The proof below shows that these extensions satisfy $f+g \in \mathcal{L}^p(A;\mu)$.

  Thus it makes sense to write $[f+g]$ for $f, g \in \mathcal{L}^p(A;\mu)$, and moreover,
  $$[f]+[g] := [f+g] \quad \text{for } [f],[g] \in L^p(A;\mu)$$
  is well-defined.

Proof of Theorem 6.10.

$$\boxed{\text{STEP } 1: \ [f], [g] \in L^p(A;\mu) \Rightarrow [f+g] \in L^p(A;\mu)}$$

- Given $[f], [g] \in L^p(A;\mu)$, we can choose $\mathbb{R}$-valued representatives $f, g \in \mathcal{L}^p(A;\mu)$. Then:

  For $1 \leq p < \infty$:

  $$|f(x)+g(x)|^p \leq [|f(x)|+|g(x)|]^p \leq [2 \max\{|f(x)|, |g(x)|\}]^p$$
  $$\leq 2^p (|f(x)|^p + |g(x)|^p),$$

  thus $f+g \in \mathcal{L}^p(A;\mu)$, and hence $[f+g] =: [f]+[g] \in L^p(A;\mu)$.

  For $p = \infty$:

  $$|f(x)+g(x)| \leq |f(x)|+|g(x)| \leq \|f\|_\infty + \|g\|_\infty \quad \mu \text{ a.e. } x \in A,$$

  recall the comment below Def. 6.4,

  thus

  $$\|f+g\|_\infty := \inf\{c \geq 0 : |f(x)+g(x)| \leq c \ \ \mu \text{ a.e. } x \in A\}$$
  $$\leq \|f\|_\infty + \|g\|_\infty$$

  and hence $f+g \in \mathcal{L}^\infty(A;\mu)$ and $[f+g] =: [f]+[g] \in L^\infty(A;\mu)$.

$\boxed{\text{STEP } 2: \quad \|f+g\|_p \leq \|f\|_p + \|g\|_p \text{ for } f, g \in L^p(A)}$

- The arguments in $\boxed{\text{STEP } 1}$ immediately yield the inequality in the case $p \in \{1, \infty\}$.

- If $1 < p < \infty$, we apply Hölder's inequality as planned:

   - $$\|f+g\|_p^p = \int_A |f+g|^p \, d\mu$$

     $$\leq \int_A |f| \, |f+g|^{p-1} d\mu + \int_A |g| \, |f+g|^{p-1} d\mu$$

     $$\overset{[\text{Hölder}]}{\leq} \left(\int_A |f|^p d\mu \right)^{\frac{1}{p}} \left(\int_A |f+g|^p d\mu \right)^{\frac{p-1}{p}}$$

     $$+ \left(\int_A |g|^p d\mu \right)^{\frac{1}{p}} \left(\int_A |f+g|^{p-1} d\mu \right)^{\frac{p-1}{p}}$$

     $$= \|f\|_p \, \|f+g\|_p^{p-1} + \|g\|_p \, \|f+g\|_p^{p-1}.$$

   - Without loss of generality: $\|f+g\|_p \neq 0$ (otherwise the inequality is trivial).

     Moreover, by $\boxed{\text{STEP } 1}$: $\|f+g\|_p < \infty$

     Hence, dividing by $\|f+g\|_p^{p-1}$, we obtain from the above inequality:

     $$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

$\square$