Geometric Measure Theory (MATS2110) Exercises 5

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Deadline: Tue 29.4.2025, 12:15

Rectifiable and purely unrectifiable sets; Besicovitch's projection theorem

1. [hand in] Elementary properties of rectifiable sets

Prove the following elementary properties of rectifiable sets in $\mathbb{R}^d$:

1.a (0.3 point) Every subset of a k-rectifiable set is k-rectifiable.

1.b (0.3 point) The countable union of k-rectifiable sets is k-rectifiable.

1.c (0.4 point) If E is k-rectifiable, there is a k-rectifiable Borel set $B \supseteq E$ with $\mathcal{H}^k(B) = \mathcal{H}^k(E)$.

2. [demo] Hausdorff density of rectifiable curves

Let Γ be a rectifiable curve in $\mathbb{R}^d$. Verify directly (without referring to more general results from rectifiability theory) that the 1-dimensional density of Γ exists and satisfies $\Theta^1(\Gamma, x) = 1$ for $\mathcal{H}^1$ almost every $x \in \Gamma$.

3. [demo] Unrectifiability of the 4-corner Cantor set – I

Let $K = C(1/4) \times C(1/4) \subset \mathbb{R}^2$ be the 4-corner Cantor set as defined in the first lecture. Prove that K does not possess an approximate tangent at any of its points.

By the result stated in Lecture 5.I, this shows that K is purely 1-unrectifiable.

4. [demo] Unrectifiability of the 4-corner Cantor set – II

3.a (0.8 point) Let A be a $\mathcal{H}^1$-measurable subset of a rectifiable curve Γ in $\mathbb{R}^2$. Show that if $\mathcal{H}^1(A) > 0$ and $\mathcal{H}^1(\pi_e(A)) = 0 = \mathcal{H}^1(\pi_{e'}(A))$ for two distinct $e, e' \in S^1$, then $e = -e'$.

Hint using densities (there are other possible proofs): Pick a point $x \in A$ with $\Theta^1(A, x) = \Theta^1(\Gamma, x) = 1$. (Why does such a point exist? You may use the result from Problem 2.) Use the information on the densities to find for every $\varepsilon > 0$ a radius $r > 0$ such that $\mathcal{H}^1((\Gamma \setminus A) \cap B(x, r)) < \varepsilon \mathcal{H}^1(\Gamma \cap B(x, r))$. Choose a suitable arc $\Gamma_0 \subset \Gamma \cap B(x, r)$ to show that $\mathcal{H}^1(\pi_e(A)) > 0$ except possibly for a set of directions contained in a circular arc of length $\sim \varepsilon$ (here e and $-e$ are said to have the same "direction").

3.b (0.2 point) Use 3.a to give an alternative proof for the pure 1-unrectifiability of the 4-corner Cantor set $K = C(1/4) \times C(1/4)$.

5. [hand in] Covering by a single Lipschitz image

Let $E \subset \mathbb{R}^d$ be a set of the form $E = \bigcup_{i \in \mathbb{N}} f_i(A_i)$, where for each i, the function $f_i : A_i \to \mathbb{R}^d$ is Lipschitz and defined on a bounded set $A_i \subset \mathbb{R}^k$. Show that there exists a (not necessarily bounded) set $A \subset \mathbb{R}^k$ and a Lipschitz function $f : A \to \mathbb{R}^k$ such that $f(A) = E$.

6. [hand in] (Bonus exercise; 0.5 bonus points)

Submit a 1-page summary of what happened in the previous two lectures of the course. The format is free: the summary can contain figures, lists, mind maps etc. Was there something you found surprising or confusing? Are there questions you would like to study further? Make the summary useful for yourself and do not just copy verbatim from the lecture notes! If you have submitted a summary, be prepared to discuss it during the demo session.